Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Approximation Theory 129 (2004) 28–57

JOURNAL OF Approximation Theory

http://www.elsevier.com/locate/jat

A Duchon framework for the sphere

Simon Hubbert[a,*] and Tanya M. Morton[b]

[a] *Lehrstuhl für Numerische Mathematik, Justus-Liebig-Universität, Heinrich-Buff-Ring 44, Giessen 35392, Germany*

[b] *The Mathworks Ltd, Matrix House, Cowley Park, Cambridge CB4 0HH, UK*

Received 14 August 2002; accepted in revised form 27 April 2004

Communicated by Robert Schaback

Abstract

In his fundamental paper (RAIRO Anal. Numer. 12 (1978) 325) Duchon presented a strategy for analysing the accuracy of surface spline interpolants to sufficiently smooth target functions. In the mid-1990s Duchon's strategy was revisited by Light and Wayne (J. Approx. Theory 92 (1992) 245) and Wendland (in: A. Le Méhauté, C. Rabut, L.L. Schumaker (Eds.), Surface Fitting and Multiresolution Methods, Vanderbilt Univ. Press, Nashville, 1997, pp. 337–344), who successfully used it to provide useful error estimates for radial basis function interpolation in Euclidean space. A relatively new and closely related area of interest is to investigate how well radial basis functions interpolate data which are restricted to the surface of a unit sphere. In this paper we present a modified version Duchon's strategy for the sphere; this is used in our follow up paper (L_p-error estimates for radial basis function interpolation on the sphere, preprint, 2002) to provide new L_p error estimates ($p \in [1, \infty]$) for radial basis function interpolation on the sphere.

Keywords: Approximation theory; Spherical Fourier theory; Radial basis functions

*Corresponding author. Fax: +49-641-99-32199.
E-mail address: simon.hubbert@math.uni-giessen.de (S. Hubbert).

0021-9045/$ - see front matter
doi:10.1016/j.jat.2004.04.005

1. Introduction

In the Euclidean space setting the so-called Duchon framework [3] is a well-known and useful strategy for providing error bounds for radial basis function interpolation [6,11]. The approach itself is a local-global strategy whereby a function defined on a suitable domain $\Omega \subset \mathbb{R}^d$ is examined locally over a collection of open Euclidean balls B_i whose union covers Ω. When applied to interpolation problems, the overall strategy allows us to glue together local interpolation error estimates (over the Euclidean balls) and so provide a useful global error bound. In this paper we specialise the framework to the $(d-1)$-dimensional unit sphere, $S^{d-1} \subset \mathbb{R}^d$. In this setting our local analysis will take place on a geodesic ball

$$G(z,\theta) = \{\xi \in S^{d-1} : g(z,\xi) < \theta\}, \quad z \in S^{d-1} \;\; \theta \in (0, 2\pi),$$

where $g : S^{d-1} \times S^{d-1} \to [0,\pi]$ denotes the geodesic distance defined by

$$g(\xi,\eta) = \cos^{-1}(\xi^T \eta), \quad \xi,\eta \in S^{d-1}. \tag{1.1}$$

Our purpose is two fold. First, we will show that it is possible to cover S^{d-1} with a finite collection of geodesic balls $G_i = G(z_i,\theta)$ each with the same radius θ, such that for any f belonging to the Sobolev space $W_2^\beta(S^{d-1})$, we have

$$\sum_{G_i} ||f|_{G_i}||^2_{W_2^\beta(G_i)} \leqslant Q||f||^2_{W_2^\beta(S^{d-1})}, \tag{1.2}$$

where the constant Q is independent of θ. The idea of considering *local* restrictions $f|_{G(z,\theta)}$ of a *global* function $f \in W_2^\beta(S^{d-1})$ is key to the success of Duchon's strategy for the sphere [5]. This brings us to our second purpose which is, first of all, to construct a linear extension operator

$$E : W_2^\beta(G(z,\theta)) \to W_2^\beta(S^{d-1})$$

which satisfies $Ef|_{G(z,\theta)} = f$ and

$$||Ef||_{W_2^\beta(S^{d-1})} \leqslant \mathcal{K}||f||_{W_2^\beta(G(z,\theta))}, \quad \text{for all } f \in W_2^\beta(G(z,\theta)),$$

where the constant $\mathcal{K}$ is independent of f. We remark that this extension result is essentially known, indeed such results hold true on compact boundary free Riemannian manifolds (see [7]). However, it is instructive to run through the construction details for the sphere because we can then easily establish then that the extension constant $\mathcal{K}$ necessarily depends on the radius θ of the geodesic ball. The dependence of $\mathcal{K}$ on θ is unavoidable. However, if we impose a set of dense zero conditions on the functions, that is we let $\Xi = \{\xi_i\}_{i=1}^N$ denote a set of distinct points in $G(z,\theta)$ and consider the subspace

$$\widetilde{W}_2^\beta(G(z,\theta)) = \{f \in W_2^\beta(G(z,\theta)) : f(\xi) = 0, \;\; \xi \in \Xi\},$$

then

$$||Ef||_{W_2^\beta(S^{d-1})} \leqslant \widetilde{\mathcal{K}}||f||_{W_2^\beta(G(z,\theta))}, \quad \text{for all } f \in \widetilde{W}_2^\beta(G(z,\theta)),$$

where $\widetilde{\mathcal{K}}$ is independent of θ. With this established the specialization of the Duchon framework to the sphere is complete.

1.1. Interpolation theory of Banach spaces: basic results

Let A_0 and A_1 be Banach spaces such that there is a continuous inclusion $A_1 \subset A_0$. Two such Banach spaces are said to be an *interpolation pair* (A_0, A_1). For any $f \in A_0$ and $t > 0$, we define the K-functional as

$$K(t,f) = \inf_{g \in A_1} (||f - g||_{A_0} + t||g||_{A_1}). \tag{1.3}$$

For $\tau \in (0, 1)$, the *interpolation space* $A_\tau = (A_0, A_1)_\tau$ is defined to be the Banach subspace of A_0 for which the following norm is finite:

$$||f||_\tau = ||K(t,f)t^{-\tau}||_{L_2((0,\infty),\frac{dt}{t})} = \left(\int_0^\infty \left(\frac{K(t,f)}{t^\tau}\right)^2 \frac{dt}{t}\right)^{1/2}. \tag{1.4}$$

Operator interpolation property. Suppose that we have two interpolation pairs (A_0, A_1) and (B_0, B_1) as above, and a linear operator T that maps A_i to B_i, such that

$$||Tf||_{B_i} \leqslant C_i \cdot ||f||_{A_i}, \quad \text{for all } f \in A_i, \;\; i \in \{0, 1\}.$$

Then the *operator interpolation property* says that T may be viewed as a bounded linear map of A_τ to B_τ and

$$||Tf||_{B_\tau} \leqslant C_0^{1-\tau} C_1^\tau \cdot ||f||_{A_\tau}, \quad \text{for all } f \in A_\tau. \tag{1.5}$$

The above results may be found in [10]. Our main interest lies in applying these results to the Sobolev spaces $W_2^k(\Omega)$, where k (the order of the Sobolev space) is a non-negative integer and Ω is a bounded open set in $\mathbb{R}^d$. Specifically, $W_2^k(\Omega)$ is defined to be the Hilbert subspace of functions $f \in L_2(\Omega)$ for which the following norm is finite:

$$||f||_{W_2^k(\Omega)} = (f,f)^{1/2}_{W_2^k(\Omega)} = \left(\sum_{0 \leqslant |\alpha| \leqslant k} ||D^\alpha f||^2_{L_2(\Omega)}\right)^{\frac{1}{2}}. \tag{1.6}$$

Due to the demand on the derivatives in (1.6) it is clear that $W_2^m(\Omega) \subset W_2^k(\Omega)$ for $0 \leqslant k < m$, and so $(W_2^m(\Omega), W_2^k(\Omega))$ is an interpolation pair. The importance of interpolation spaces in this case comes from the fact that

$$(W_2^k(\Omega), W_2^m(\Omega))_\tau \cong W_2^{(1-\tau)k+\tau m}(\Omega), \quad \text{whenever } (1-\tau)k + \tau m \in \mathbb{N},$$

where $\cong$ denotes norm equivalence; see [7]. For this reason, the fractional Sobolev space $W_2^{k+\tau}(\Omega)$ is defined as follows:

$$W_2^{k+\tau}(\Omega) = (W_2^k(\Omega), W_2^{k+1}(\Omega))_\tau = \{f \in W_2^k(\Omega) : ||f||_{W_2^{k+\tau}(\Omega)} < \infty\}, \tag{1.7}$$

where, by (1.4), we have

$$||f||_{W_2^{k+\tau}(\Omega)} = \left(\int_0^\infty \left(\frac{K(t,f)}{t^\tau} \right)^2 \frac{dt}{t} \right)^{1/2} \tag{1.8}$$

and

$$K(t,f) = \inf_{g \in W_2^{k+1}(\Omega)} (||f-g||_{W_2^k(\Omega)} + t||g||_{W_2^{k+1}(\Omega)}). \tag{1.9}$$

The development of Sobolev space theory begins with a study of the global spaces, where $\Omega = \mathbb{R}^d$. In order to generalise the various results established for $\mathbb{R}^d$ to the case of a bounded domain Ω, it is important to know whether there exists a continuous linear extension operator

$$E : W_2^k(\Omega) \to W_2^k(\mathbb{R}^d), \quad \text{satisfying} \quad (Ef)|_\Omega = f, \text{ for all } f \in W_2^k(\Omega). \tag{1.10}$$

In [9], Stein proved the following remarkable theorem.

Theorem 1.1. *Let Ω be a bounded open connected set with sufficiently smooth boundary. There exists an extension operator* (1.10) *defined for all non-negative integers k, such that*

$$||Ef||_{W_2^k(\mathbb{R}^n)} \leqslant \mathcal{C}_{\text{ext}} \cdot ||f||_{W_2^k(\Omega)}, \text{ where } \mathcal{C}_{\text{ext}} \text{ is independent of } f.$$

Furthermore, if $\Omega = B(x,r)$ is an open ball, then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ E can be chosen so that the support of Ef is contained in $B(x,(1+\varepsilon)r)$.

For an excellent overall account of Sobolev space theory see [1], Further, for a shorter but detailed review of the material needed in this paper see [7, Chapter 2].

2. Sobolev spaces on the sphere

In order to construct a Sobolev extension operator for the sphere we must, first of all, define the relevant local and global spherical Sobolev spaces. There are several (equivalent) ways of defining these spaces, however the definition that we shall use relies on the fact that the sphere is a $(d-1)$-dimensional differentiable manifold. The notion of defining a Sobolev space on a differentiable manifold was considered in [7], we shall give an account of this theory before specialising it to the sphere.

2.1. *Differentiable manifolds*

Let $\mathbb{M}$ denote a $(d-1)$-dimensional compact differentiable manifold, and suppose that $\mathcal{A} = \{U_i, \phi_i\}_{i=1}^n$ is an atlas for $\mathbb{M}$, i.e., a finite collection of *charts* (U_i, ϕ_i), where U_i are open subsets of $\mathbb{M}$, covering $\mathbb{M}$, and where ϕ_i are infinitely differentiable mappings $\phi_i : U_i \to B(0,1) \subset \mathbb{R}^{d-1}$, whose inverses ϕ_i^{-1} are also infinitely differentiable. Also, let $\{\chi_i : \mathbb{M} \to \mathbb{R}\}_{i=1}^n$ be a *partition of unity* subordinated to the atlas, i.e., a set of infinitely differentiable functions χ_i on $\mathbb{M}$ vanishing outside of compact subsets of the U_i, such that $\sum_i \chi_i = 1$.

For any function $f : \mathbb{M} \to \mathbb{R}$, we can use a partition of unity to write

$$f = \sum_{i=1}^{n} (\chi_i f), \quad \text{where } (\chi_i f)(m) = \chi_i(m) f(m), \quad m \in \mathbb{M}. \tag{2.1}$$

This gives us a decomposition of f in terms of local functions $\chi_i f$, which are compactly supported in U_i. For any function $f : \mathbb{M} \to \mathbb{R}$ with compact support in U_i, we can define its projection $\pi_i(f) : \mathbb{R}^{d-1} \to \mathbb{R}$ onto $\mathbb{R}^{d-1}$ by

$$\pi_i(f)(x) = \begin{cases} f \circ \phi_i^{-1}(x) & \text{if } x \in B(0,1), \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

With this in place, we define the Sobolev space $W_2^\beta(\mathbb{M})$ $(\beta > 0)$ to be the set

$$W_2^\beta(\mathbb{M}) := \{ f \in L_2(\mathbb{M}) : \pi_i(\chi_i f) \in W_2^\beta(\mathbb{R}^{d-1}), \quad \text{for } i = 1, \ldots, n \}, \tag{2.3}$$

which is equipped with the norm

$$||f||_{W_2^\beta(\mathbb{M})} = \left(\sum_{i=1}^{n} ||\pi_i(\chi_i f)||^2_{W_2^\beta(\mathbb{R}^{d-1})} \right)^{\frac{1}{2}}. \tag{2.4}$$

2.2. *Application to the sphere*

Let $\hat{n} = (0, \ldots, 1)$ and $\hat{s} = (0, \ldots, -1)$ denote the north and south poles of the S^{d-1}, respectively. Then a simple open cover for the sphere is provided by

$$U_1 = G(\hat{n}, \theta_0) \quad \text{and} \quad U_2 = G(\hat{s}, \theta_0), \text{ where } \quad \theta_0 \in \left(\frac{\pi}{2}, \frac{2\pi}{3} \right). \tag{2.5}$$

Definition 2.1. The stereographic projection $\sigma_{\hat{n}}$ of the punctured sphere $S^{d-1} \backslash \{\hat{n}\}$ onto $\mathbb{R}^{d-1}$ is defined as the mapping that takes $\xi \in S^{d-1} \backslash \{\hat{n}\}$ to the intersection of the equatorial hyperplane $\{\xi_d = 0\}$, and the extended line that passes through ξ and $\hat{n}$.

We remark that the stereographic projection $\sigma_{\hat{s}}$ based on $\hat{s}$ can be defined analogously. Using elementary trigonometry we can set

$$\phi_1 = \frac{1}{\tan(\theta_0/2)} \cdot \sigma_{\hat{s}} \quad \text{and} \quad \phi_2 = \frac{1}{\tan(\theta_0/2)} \cdot \sigma_{\hat{n}}, \tag{2.6}$$

and conclude that $\mathcal{A} = \{U_i, \phi_i\}_{i=1}^2$ is a C^∞ atlas of covering coordinate charts for the sphere. Hence, S^{d-1} is a $(d-1)$-dimensional differentiable manifold and so we define the Sobolev space $W_2^\beta(S^{d-1})$ to be the set

$$\{ f \in L_2(S^{d-1}) : \pi_i(\chi_i f) \in W_2^k(\mathbb{R}^{d-1}) \quad \text{for } i = 1, 2 \}, \tag{2.7}$$

which is equipped with the norm

$$||f||_{W_2^\beta(S^{d-1})} = \left(\sum_{i=1}^{2} ||\pi_i(\chi_i f)||^2_{W_2^\beta(\mathbb{R}^{d-1})} \right)^{\frac{1}{2}}. \tag{2.8}$$

This definition seems to depend on the choice of atlas used to define S^{d-1}. However, it can be shown that any two spaces defined using two different atlases coincide as sets, and norms (2.8) are equivalent, see [7] for details.

In order to define the spaces on some geodesic ball, $G(z,\theta)$, we use the coordinate charts to specify open sets in $\mathbb{R}^{d-1}$ by

$$\Omega_i = \phi_i(G(z,\theta) \cap U_i) \quad \text{for } i \in \{1,2\}. \tag{2.9}$$

The local Sobolev space $W_2^\beta(G(z,\theta))$ of order β is defined to be the set

$$\{f \in L_2(G(z,\theta)) : \pi_i(\chi_i f)|_{\Omega_i} \in W_2^\beta(\Omega_i) \text{ for } i \in \{1,2\}, \Omega_i \neq \emptyset\}, \tag{2.10}$$

which is equipped with the norm

$$||f||_{W_2^\beta(G(z,\theta))} = \left(\sum_{i=1}^{2} ||\pi_i(\chi_i f)|_{\Omega_i}||^2_{W_2^\beta(\Omega_i)} \right)^{\frac{1}{2}} \tag{2.11}$$

where, if $\Omega_i = \emptyset$, then we adopt the convention that $||\cdot||_{W_2^\beta(\Omega_i)} = 0$.

2.3. On specific local Sobolev spaces

Let $\mathcal{A} = \{U_i, \phi_i\}_{i=1}^2$ denote a fixed atlas for S^{d-1} and let $\{\chi_i\}_{i=1}^2$ denote a corresponding partition of unity. Our aim here is to present a closer analysis of the local spaces $W_2^k(G(z,\theta))$ where k is a non-negative integer.

For any function $f \in W_2^k(G(z,\theta))$, we can use the partition of unity and the atlas to write

$$f = \sum_{i=1}^{2} (\chi_i f)|_{G(z,\theta) \cap U_i} = \sum_{i=1}^{2} (\chi_i f) \circ \phi_i^{-1}|_{\Omega_i = \phi_i(G(z,\theta) \cap U_i)}. \tag{2.12}$$

Further, we have the following useful observation.

Observation 2.2. *Each χ_i has compact support*, $\text{supp}\{\chi_i\} \subset U_i$. *Thus, there exists a positive constant $C_{\mathcal{A}}$, depending only on $\mathcal{A}$ and the partition of unity $\{\chi_1, \chi_2\}$, such that the geodesic distance of* $\text{supp}\{\chi_i\}$ *from the boundary of U_i is strictly greater than $C_{\mathcal{A}}$, for $i \in \{1,2\}$.*

Let $\alpha \in (0,1)$ and let $V_i(\alpha)$ denote the $\alpha C_{\mathcal{A}}$-geodesic neighbourhood of $\text{supp}\{\chi_i\}$, for $i \in \{1,2\}$. Furthermore, if $\theta < C_{\mathcal{A}}/3$ and $z \in S^{d-1}$, then we have the following cases (Fig. 1):

1. $z \notin \overline{V_2(1/3)} \Rightarrow \overline{G(z,\theta)} \subset V_1(2/3) \subset U_1$, and $\text{supp}\{\chi_2\} \cap G(z,\theta) = \emptyset$,

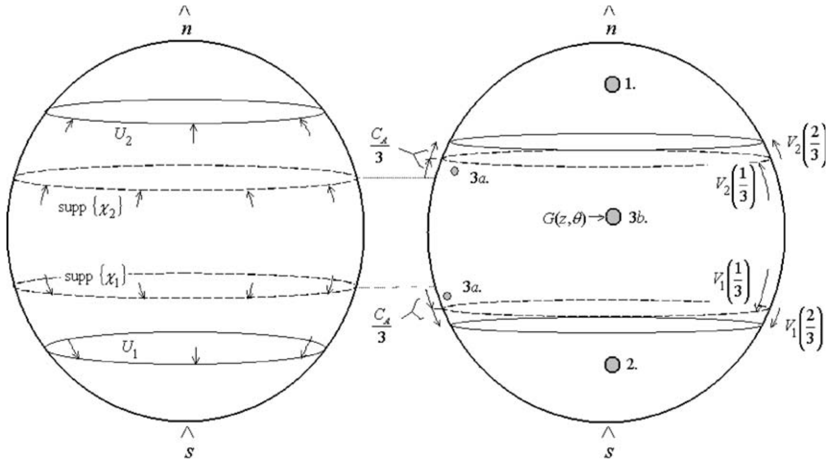

Fig. 1. To illustrate the positioning of a geodesic ball of radius $\theta < C_{\mathcal{A}}/3$.

2. $z \notin \overline{V_1(1/3)} \Rightarrow \overline{G(z,\theta)} \subset V_2(2/3) \subset U_2$, and $\text{supp}\{\chi_1\} \cap G(z,\theta) = \emptyset$,
3. $z \in \overline{V_1(1/3)} \cap \overline{V_2(1/3)} \Rightarrow \overline{G(z,\theta)} \subset V_i(2/3) \subset U_i$, for $i \in \{1,2\}$, and
   (a) either $\text{supp}\{\chi_1\} \cap G(z,\theta)$ or $\text{supp}\{\chi_2\} \cap G(z,\theta)$ is non-empty,
   (b) both $\text{supp}\{\chi_1\} \cap G(z,\theta)$ and $\text{supp}\{\chi_2\} \cap G(z,\theta)$ are non-empty.

The condition $\theta < C_{\mathcal{A}}/3$ is sufficient to guarantee that closure of $G(z,\theta)$ is a subset of at least one of the open subsets U_1 or U_2, defined by (2.5). It is well-known (see [8]) that the stereographic coordinate charts $\{\phi_i\}_{i=1}^2$ as defined in Eq. (2.6) map geodesic balls to Euclidean balls, thus we can deduce that

$$\phi_i(G(z,\theta)) = \Omega_i = B(x_i, r_i) \quad \text{and} \quad \phi_i(G(z, C_{\mathcal{A}}/3)) = B(x_i^{\mathcal{A}}, r_i^{\mathcal{A}}). \tag{2.13}$$

In general $x_i \neq x_i^{\mathcal{A}}$, that is, concentric geodesic balls are not, in general, mapped to concentric Euclidean balls.

For further illustration, suppose that $z \in S^{d-1}$ is positioned as in case 3 above. In this case we have the following strict inclusions:

$$\overline{G(z,\theta)} \subset \overline{G(z, C_{\mathcal{A}}/3)} \subset V_i(2/3) \subset U_i, \quad i \in \{1,2\},$$

and thus both Ω_1 and Ω_2 are Euclidean balls. Indeed, taking the ϕ_i images and using (2.13) gives

$$\Omega_i \subset \underbrace{\overline{\phi_i(G(z,\theta))}}_{=\overline{B(x_i,r_i)}} \subset \underbrace{\overline{\phi_i(G(z,C_{\mathcal{A}}/3))}}_{=\overline{B(x_i^{\mathcal{A}},r_i^{\mathcal{A}})}} \subset \phi_i(V_i(2/3)) \subset B(0,1), \quad i \in \{1,2\}.$$

Since these inclusions are strict there exists a positive constant $e_{\mathcal{A}}$ such that

$$\overline{B(x_i^{\mathcal{A}}, r_i^{\mathcal{A}})} \subset \phi_i(V_i(2/3)) \subset B(0, 1 - e_{\mathcal{A}}) \subset B(0,1), \quad i \in \{1,2\}.$$

Thus, for any positive $\varepsilon < e_{\mathcal{A}}$, we have

$$\Omega_i = B(x_i, r_i) \subset \overline{\phi_i(G(z,\theta))} \subset B(x_i, r_i + \varepsilon) \subset B(x_i^{\mathcal{A}}, r_i^{\mathcal{A}} + \varepsilon) \subset B(0,1),$$
$$i \in \{1, 2\}. \tag{2.14}$$

Also, when $\overline{G(z,\theta)}$ is *not* completely contained in one of the U_i (a possibility covered by cases 1 and 2 above) then we note that $\text{supp}\{\chi_i\} \cap G(z,\theta) = \emptyset$. In this case, for any given $f \in W_2^k(G(z,\theta))$, we can deduce that

$$f_i = (\chi_i f) \circ \phi_i^{-1}|_{\Omega_i} = (\chi_i f)|_{G(z,\theta) \cap U_i} = 0. \tag{2.15}$$

In summary we have (see Fig. 2):

Lemma 2.3. *Let $C_{\mathcal{A}}$ be as in Observation 2.2. Then for any $z \in S^{d-1}$ and $\theta < C_{\mathcal{A}}/3$ we have*

(i) *at least one of the open sets Ω_i $(i \in \{1, 2\})$ as defined in Eq.* (2.9), *is an open Euclidean ball, $B(x_i, r_i)$;*
(ii) *there exists a positive constant $e_{\mathcal{A}}$, depending only on the atlas $\mathcal{A}$, such that, if $\Omega_i = B(x_i, r_i)$, then $B(x_i, r_i + \varepsilon) \subset B(0,1)$ for all $0 < \varepsilon < e_{\mathcal{A}}$;*
(iii) *if Ω_i is not an open Euclidean ball and $f \in W_2^k(G(z,\theta))$ then $(\chi_i f) \circ \phi_i^{-1}|_{\Omega_i} = 0$.*

Lemma 2.3 allows us to take the view that a Sobolev space on a suitable geodesic ball in S^{d-1} essentially "behaves" like a Sobolev space on a Euclidean ball in $\mathbb{R}^{d-1}$. The following result shows that the radii of the geodesic and Euclidean balls are comparable.

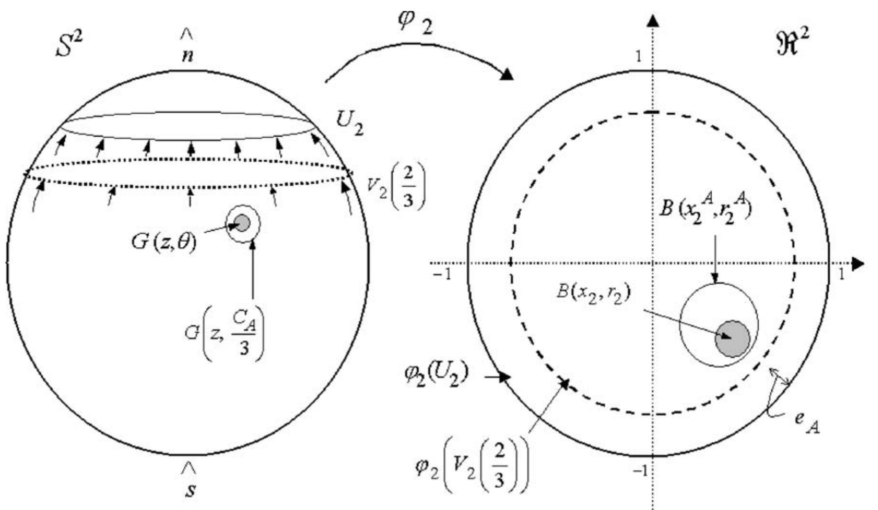

Fig. 2. Illustration of Lemma 2.3.

Lemma 2.4. *Assume that* $\overline{G(z,\theta)} \subset U_i,\ i \in \{1,2\}$, *and let* $B(x_i, r_i)$ *be as in* (2.13). *Then there exist positive constants* c_0 *and* C_0 *such that*

$$c_0 \cdot \theta \leqslant r_i \leqslant C_0 \cdot \theta, \quad i \in \{1,2\}. \tag{2.16}$$

Proof. The Euclidean and geodesic distances (1.1) between any two points $\xi, \eta \in S^{d-1}$ are related by the formula

$$d(\xi,\eta) = ||\xi - \eta|| = 2\sin\left(\frac{g(\xi,\eta)}{2}\right).$$

Furthermore, if $\xi, \eta \in S^{d-1} \backslash \{\hat{n}\}$, we have the following relationship from [8]:

$$||\xi - \eta|| = \frac{2||\sigma_{\hat{s}}(\xi) - \sigma_{\hat{s}}(\eta)||}{(1 + ||\sigma_{\hat{s}}(\xi)||^2)^{1/2}(1 + ||\sigma_{\hat{s}}(\eta)||^2)^{1/2}}.$$

We remark that the analogous relation holds for $\xi, \eta \in S^{d-1} \backslash \{\hat{s}\}$. Let $\xi, \eta \in G(z,\theta)$ then, without loss, we shall establish (2.16) for $i = 1$. The above relations and (2.6) yield

$$\sin\left(\frac{g(\xi,\eta)}{2}\right) = \frac{\tan(\theta_0/2)||\phi_1(\xi) - \phi_1(\eta)||}{(1 + \tan^2(\theta_0/2)||\phi_1(\xi)||^2)^{1/2}(1 + \tan^2(\theta_0/2)||\phi_1(\eta)||^2)^{1/2}}.$$

Since $\phi_1(U_1) = B(0,1)$, we can maximise and minimise this expression by assuming that $||\phi_1(\xi)|| = ||\phi_1(\eta)||$ equals 0 and 1, respectively. This gives

$$\sin\theta_0||\phi_1(\xi) - \phi_1(\eta)|| \leqslant 2\sin\left(\frac{g(\xi,\eta)}{2}\right) \leqslant 2\tan(\theta_0/2)||\phi_1(\xi) - \phi_1(\eta)||.$$

For any $\alpha \in (0, \pi/3)$, the small angle result, $\alpha/2 \leqslant \sin\alpha \leqslant \alpha$, implies that

$$\sin\theta_0||\phi_1(\xi) - \phi_1(\eta)|| \leqslant g(\xi,\eta) \leqslant 2\theta$$

and

$$g(\xi,\eta) \leqslant 4\tan(\theta_0/2)||\phi_1(\xi) - \phi_1(\eta)|| \leqslant 8\tan(\theta_0/2)r_1.$$

Since we can write

$$2r_1 = \sup_{\xi,\eta \in G(z,\theta)} ||\phi_1(\xi) - \phi_1(\eta)|| \quad \text{and} \quad 2\theta = \sup_{\xi,\eta \in G(z,\theta)} g(\xi,\eta),$$

the proof is completed by taking the supremum on the left-hand sides of the two inequalities above. We find that $c_0 = (4\tan(\theta_0/2))^{-1}$ and $C_0 = (\sin\theta_0)^{-1}$. □

3. Duchon's inequality for the sphere

The original Duchon framework makes use of a scaled integer lattice in $\mathbb{R}^d$ to provide a regular mesh with a specified spacing. While we do not have quite such a regular mesh on the sphere, we can find a quasi uniform mesh that will satisfy our requirement. An example can be obtained by inscribing a d-dimensional cube inside

S^{d-1} with a scaled integer lattice embedded on each side, then radially projecting the lattice points to S^{d-1}.

Lemma 3.1. *Let $d \geqslant 2$, be an integer and set*

$$M = 2\sqrt{d-1} \quad \text{and} \quad \delta_d = \frac{1}{4d^{3/2}}.$$

Let M_1 be an arbitrary positive number, $\theta \in (0, \pi/3)$ and set

$$h_0 := \frac{\theta}{M + M_1 + \delta_d}. \tag{3.1}$$

Then, for any $h \in (0, h_0)$, there exists a set of points $Z_h \subset S^{d-1}$ such that

$$S^{d-1} = \bigcup_{z \in Z_h} G(z, Mh).$$

Let F_A denote the characteristic function of a set $A \subset S^{d-1}$. There exists a positive integer Q independent of h such that

$$\sum_{z \in Z_h} F_{G(z, \bar{M}h)} \leqslant Q, \quad \text{where } \bar{M} = M + M_1. \tag{3.2}$$

Further, the cardinality of Z_h is bounded above by $C_Q h^{-(d-1)}$, where C_Q is independent of h.

Proof. Let M_1 be a given positive constant, $\theta \in (0, \pi/3)$ and let h_0 be as in (3.1) and choose $h \in (0, h_0)$. To specify a mesh Z_h for the sphere we inscribe a d-dimensional cube inside S^{d-1}. This cube will have side $2/\sqrt{d}$. Let $n_h \geqslant 2$ denote the integer such that

$$\frac{1}{n_h} \leqslant h < \frac{1}{n_h - 1}. \tag{3.3}$$

On each face of the inscribed cube we place a regular mesh of dimension $d-1$ such that each subcube has side $2/(n_h\sqrt{d})$. That is, we place a lattice of points isomorphic to

$$\frac{2}{n_h\sqrt{d}} \cdot \mathbb{Z}^{d-1} \cap \left[-\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}\right]^{d-1}$$

on each side of the cube. We now define Z_h to be the radial projection of these points on the cube onto S^{d-1}.

Suppose that two points x_1 and x_2 lie on one side of the inscribed cube and are a distance b apart. Let ξ_1 and ξ_2 be the radial projections onto S^{d-1} of x_1 and x_2, respectively. Consider Fig. 3. The point $\hat{x}$ is the closest point to the origin from the (extended) line connecting x_1 and x_2. Further, r_1 and r_2 represent the respective distances of x_1 and x_2 relative to $\hat{x}$ (with the convention that $r_2 \geqslant 0$). We observe that

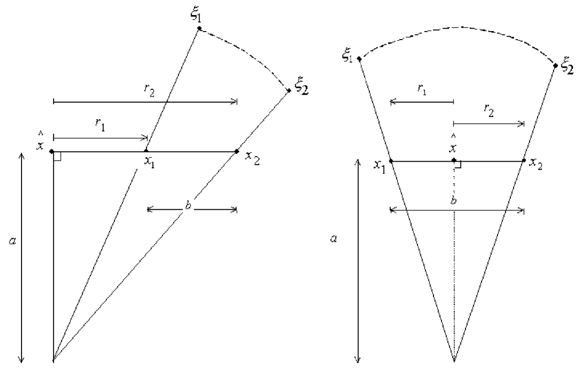

Fig. 3. Illustration of quasi-uniform mesh proof.

the distance a of $\hat{x}$ to the origin satisfies

$$\frac{1}{\sqrt{d}} \leqslant a < 1. \tag{3.4}$$

The geodesic distance $g(\xi_1, \xi_2)$ is given by

$$g(\xi_1, \xi_2) = \left| \tan^{-1} \frac{r_1}{a} - \tan^{-1} \frac{r_2}{a} \right|.$$

Employing the mean value theorem we can deduce that

$$g(\xi_1, \xi_2) = \frac{b}{a} \cdot \frac{1}{1+\xi^2}, \quad \text{for some } \xi \in \left(-\frac{1}{a}\sqrt{\frac{d-1}{d}}, \frac{1}{a}\sqrt{\frac{d-1}{d}} \right). \tag{3.5}$$

Maximising the RHS of (3.5) and using (3.4) gives $g(\xi_1, \xi_2) \leqslant \frac{b}{a} < b\sqrt{d}$. Similarly, minimising the RHS gives

$$g(\xi_1, \xi_2) \geqslant \frac{b}{a} \cdot \frac{ad^2}{d(a^2+1)-1} > \frac{b}{a} \cdot \frac{1}{2d-1} > \frac{b}{2d}.$$

Thus, we have shown

$$\frac{b}{2d} \leqslant g(\xi_1, \xi_2) \leqslant b\sqrt{d}. \tag{3.6}$$

For any $\xi \in S^{d-1}$, let z_ξ denote its closest point from Z_h. Let $\mathcal{C}^d$ denote the d-dimensional cube inscribed into S^{d-1}, then we can view the radial projection as a mapping $\text{Proj} : \mathcal{C}^d \to S^{d-1}$. In particular, let $x \in \mathcal{C}^d$ be such that

$$\text{Proj}(x) = \xi.$$

The point x necessarily lies within a $(d-1)$-dimensional subcube, which contains the lattice point $\hat{x}$ such that

$$\text{Proj}(\hat{x}) = \xi_z.$$

The furthest that x can be away from $\hat{x}$ is bounded above by $\frac{2}{n_h}\sqrt{\frac{d-1}{d}}$, the diameter of the subcube. Hence, by (3.6) and (3.3), any $\xi \in S^{d-1}$ is such that

$$\min_{z \in Z_h} g(z, \xi) = g(z_\xi, \xi) < \frac{2}{n_h}\sqrt{d-1} \leqslant 2h\sqrt{d-1}.$$

This proves the first part of the theorem for $M = 2\sqrt{d-1}$.

The minimum separation distance of the lattice points on the surface of the inscribed cube is $2/(n_h\sqrt{d})$. Therefore by (3.6) and (3.3)

$$\min_{z,z' \in Z_h} g(z, z') > \frac{1}{2}\left(\frac{2}{n_h\sqrt{d}}\right)\frac{1}{d} > \frac{1}{2}\frac{h}{d^{3/2}} = 2\delta_d h. \tag{3.7}$$

We now show that the second part of the theorem holds by an elementary surface area argument. Let $\xi \in S^{d-1}$ and suppose that

$$\sum_{z \in Z_h} F_{G(z,(M+M_1)h)}(\xi) = N,$$

that is,

$$g(\xi, z_i) < (M + M_1)h = \bar{M}h, \quad \text{for } z_i \in Z_h, \ \ i = 1, \dots, N.$$

This implies that

$$G(z_i, \delta_d h), \subset G(\xi, \bar{M}h + \delta_d h), \quad \text{for } i = 1, \dots, N. \tag{3.8}$$

For any geodesic ball $G(\xi, \theta)$, with $\xi \in S^{d-1}$ and $\theta \in (0, \pi/3)$ there exists positive constants $\mathcal{C}_1^a$ and $\mathcal{C}_2^a$, depending only on d, such that its surface area is bounded by

$$\mathcal{C}_1^a \cdot \theta^{d-1} \leqslant \int_{G(\xi,\theta)} d\omega_{d-1} \leqslant \mathcal{C}_2^a \cdot \theta^{d-1}, \quad \xi \in S^{d-1}.$$

We note that, as a consequence of (3.7), the balls $G(z_i, \delta_d h)$ $(i = 1, \dots, N)$ are disjoint. Thus, by (3.8), we can conclude that the area of $G(\xi, \bar{M}h + \delta_d h)$, which, by the choice of h_0, is bounded above by $\mathcal{C}_2^a \cdot (\bar{M}h + \delta_d h)^{d-1}$, must be at least $N\mathcal{C}_1^a \cdot (\delta_d h)^{d-1}$. This implies that

$$N\mathcal{C}_1^a \cdot \delta_d^{d-1} \leqslant \mathcal{C}_2^a \cdot (\bar{M} + \delta_d)^{d-1}.$$

Therefore, there exists an integer Q that is independent of ξ and h such that

$$N \leqslant \frac{\mathcal{C}_2^a}{\mathcal{C}_1^a}\left(\frac{\bar{M}}{\delta_d} + 1\right)^{d-1} \leqslant Q.$$

To finish the proof we let $|Z_h|$ denote the cardinality of the mesh. Then we have

$$\begin{aligned}|Z_h| \cdot \mathcal{C}_1^a(\bar{M}h)^{d-1} &= \sum_{z\in Z_h} \mathcal{C}_1^a(\bar{M}h)^{d-1} \\ &\leqslant \sum_{z\in Z_h} \int_{G(z,\bar{M}h)} d\omega_{d-1}(\xi) = \sum_{z\in Z_h} \int_{S^{d-1}} F_{G(z,\bar{M}h)}(\xi) d\omega_{d-1}(\xi) \\ &= \int_{S^{d-1}} \underbrace{\sum_{z\in Z_h} F_{G(z,\bar{M}h)}(\xi)}_{\leqslant Q} \, d\omega_{d-1}(\xi) \leqslant Q\omega_{d-1}.\end{aligned}$$

That is,

$$|Z_h| \leqslant \left(\frac{Q\omega_{d-1}}{\mathcal{C}_1^a \bar{M}^{d-1}}\right) h^{-(d-1)} = C_Q h^{-(d-1)},$$

where C_Q is independent of h. □

We are now in position to prove the main result of this section.

Theorem 3.2. *Let $d \geqslant 2$, be an integer. Let $\beta > 0$ and let M_1 be any positive number. Let h_0 be as in* (3.1) *with $\theta = C_{\mathcal{A}}/3$, that is*

$$h_0 = \frac{C_{\mathcal{A}}}{3(\bar{M} + \delta_d)} \quad \textit{where } \bar{M} = 2\sqrt{d-1} + M_1 \textit{ and } \delta_d = \frac{1}{4d^{3/2}}.$$

Let $h \in (0, h_0)$ and let Z_h denote the corresponding quasi-uniform mesh for S^{d-1} from Lemma 3.1. *Then, for any $f \in W_2^\beta(S^{d-1})$, we have*

$$\sum_{z\in Z_h} ||f||^2_{W_2^\beta(G(z,\bar{M}h))} \leqslant Q ||f||^2_{W_2^\beta(S^{d-1})}, \tag{3.9}$$

where Q is the constant (*independent of h*) *from Lemma* 3.1.

Proof. For $z \in Z_h$ and $i \in \{1, 2\}$ we shall set

$$\Omega_i(z) = \phi_i(G(z, \bar{M}h) \cap U_i) \subset B(0, 1). \tag{3.10}$$

We begin by proving the result in the integer case. Thus, for any $f \in W_2^k(S^{d-1})$, k a non-negative integer, we use (2.11) and consider

$$\sum_{z\in Z_h} ||f||^2_{W_2^k(G(z,\bar{M}h))} = \sum_{i=1}^{2} \sum_{z\in Z_h} ||\pi_i(\chi_i f)|_{\Omega_i(z)}||^2_{W_2^k(\Omega_i(z))}. \tag{3.11}$$

Let F_A denote the characteristic function of a set $A \subset \mathbb{R}^d$. For $i \in \{1, 2\}$, consider any function $g \in W_2^k(\mathbb{R}^{d-1})$. Using (3.10) and Lemma 3.1 we can write

$$\begin{aligned}\sum_{z\in Z_h} \|g|_{\Omega_i(z)}\|^2_{W_2^k(\Omega_i(z))} &= \sum_{z\in Z_h} \sum_{|\alpha|\leqslant k} \|D^\alpha g|_{\Omega_i(z)}\|^2_{L_2(\Omega_i(z))} \\ &= \sum_{|\alpha|\leqslant k} \sum_{z\in Z_h} \int_{\Omega_i(z)} ((D^\alpha g|_{\Omega_i(z)})^2(x))\, \lambda dx \\ &= \sum_{|\alpha|\leqslant k} \int_{\mathbb{R}^{d-1}} \sum_{z\in Z_h} F_{\Omega_i(z)}(x)(D^\alpha g(x))^2\, dx \\ &\leqslant Q \sum_{|\alpha|\leqslant k} \|D^\alpha g\|^2_{L_2(\mathbb{R}^{d-1})} = Q\|g\|^2_{W_2^k(\mathbb{R}^{d-1})}. \end{aligned} \tag{3.12}$$

Applying these arguments to $g = \pi_i(\chi_i f) \in W_2^k(\mathbb{R}^{d-1})$, and substituting into (3.11) provides the result for the integer order spaces.

Since $\bar{M}h < \bar{M}h_0 < C_{\mathcal{A}}/3$, we can use Lemma 2.3(iii) to conclude that if $\Omega_i(z)$ is not an open Euclidean ball then $\pi_i(\chi_i f)|_{\Omega_i(z)}$ is the zero function. In particular, for any $f \in W_2^{k+\tau}(S^{d-1})$, where $\tau \in (0, 1)$, we can write

$$\sum_{z\in Z_h} \|f\|^2_{W_2^{k+\tau}(G(z,\bar{M}h))} = \sum_{i=1}^{2} \sum_{z\in \mathcal{E}_h(i)} \|\pi_i(\chi_i f)|_{\Omega_i(z)}\|^2_{W_2^{k+\tau}(\Omega_i(z))}, \tag{3.13}$$

where

$$\mathcal{E}_h(i) = \{z \in Z_h : \Omega_i(z) \text{ is an open Euclidean ball}\}. \tag{3.14}$$

Fix $i \in \{1, 2\}$ and let $f_i = \pi_i(\chi_i f)$. Then using (1.8) we have

$$\begin{aligned}\sum_{z\in \mathcal{E}_h(i)} \|f_i|_{\Omega_i(z)}\|^2_{W_2^{k+\tau}(\Omega_i(z))} &= \sum_{z\in \mathcal{E}_h(i)} \int_0^\infty (K(t, f_i|_{\Omega_i(z)}))^2 \frac{dt}{t^{2\tau+1}} \\ &= \int_0^\infty \sum_{z\in \mathcal{E}_h(i)} (K(t, f_i|_{\Omega_i(z)}))^2 \frac{dt}{t^{2\tau+1}}. \end{aligned} \tag{3.15}$$

Since $\Omega_i(z)$ is an open Euclidean ball for $z \in \mathcal{E}_h(i)$ we have that

$$W_2^{k+1}(\Omega_i(z)) = W_2^{k+1}(\mathbb{R}^{d-1})|_{\Omega_i(z)} \quad \text{(see [2])}.$$

Furthermore, the "restriction of functions from $\mathbb{R}^{d-1}$ to $\Omega_i(z)$" can be viewed as a continuous linear operator from $W_2^{k+1}(\mathbb{R}^{d-1})$ to $W_2^{k+1}(\Omega_i(z))$. This implies that we can rewrite the K-functional as

$$K(t, f_i|_{\Omega_i(z)}) = \inf_{g\in W_2^{k+1}(R^{d-1})} (\|(f_i - g)|_{\Omega_i(z)}\|_{W_2^k(\Omega_i(z))} + t\|g|_{\Omega_i(z)}\|_{W_2^{k+1}(\Omega_i(z))}).$$

Thus, by choosing any $\tilde{g} \in W_2^{k+1}(\mathbb{R}^{d-1})$ we have the following bound

$$\sum_{z \in \mathcal{E}_h(i)} (K(t, f_i|_{\Omega_i(z)}))^2 \leqslant \sum_{z \in \mathcal{E}_h(i)} (||(f_i - \tilde{g})|_{\Omega_i(z)}||_{W_2^k(\Omega_i(z))} + t||\tilde{g}|_{\Omega_i(z)}||_{W_2^{k+1}(\Omega_i(z))})^2.$$

Expanding the square in the above inequality gives rise to two square terms and a cross term. We investigate these individually.

Square terms.

$$\sum_{z \in \mathcal{E}_h(i)} ||(f_i - \tilde{g})|_{\Omega_i(z)}||^2_{W_2^k(\Omega_i(z))} \leqslant Q \cdot ||f_i - \tilde{g}||^2_{W_2^k(\mathbb{R}^{d-1})}.$$

This follows by the integer order argument (3.12), and similarly we have

$$t^2 \sum_{z \in \mathcal{E}_h(i)} ||\tilde{g}|_{\Omega_i(z)}||^2_{W_2^{k+1}(\Omega_i(z))} \leqslant Q \cdot t^2 ||\tilde{g}||^2_{W_2^{k+1}(\mathbb{R}^{d-1})}.$$

Cross term.

$$2t \sum_{z \in \mathcal{E}_h(i)} ||(f_i - \tilde{g})|_{\Omega_i(z)}||_{W_2^k(\Omega_i(z))} ||\tilde{g}|_{\Omega_i(z)}||_{W_2^{k+1}(\Omega_i(z))}$$
$$\leqslant 2t \left(\sum_{z \in \mathcal{E}_h(i)} ||(f_i - \tilde{g})|_{\Omega_i(z)}||^2_{W_2^k(\Omega_i(z))} \sum_{z \in \mathcal{E}_h(i)} ||\tilde{g}|_{\Omega_i(z)}||^2_{W_2^{k+1}(\Omega_i(z))} \right)^{1/2}$$
$$\leqslant Q \cdot 2t ||f_i - \tilde{g}||_{W_2^k(\mathbb{R}^{d-1})} ||\tilde{g}||_{W_2^{k+1}(\mathbb{R}^{d-1})}.$$

This follows by applying the Cauchy–Schwarz inequality and then employing integer order argument (3.12). Piecing these individual bounds together allows us to conclude that

$$\left(\sum_{z \in \mathcal{E}_h(i)} (K(t, f_i|_{\Omega_i(z)}))^2 \right)^{1/2} \leqslant Q^{1/2} (||f_i - \tilde{g}||_{W_2^k(\mathbb{R}^{d-1})} + t||\tilde{g}||_{W_2^{k+1}(\mathbb{R}^{d-1})}).$$

Taking the infimum over $\tilde{g} \in W_2^{k+1}(\mathbb{R}^{d-1})$ allows us to deduce that

$$\sum_{z \in \mathcal{E}_h(i)} (K(t, f_i|_{\Omega_i(z)}))^2 \leqslant Q \cdot K(t, f_i)^2. \tag{3.16}$$

Substituting (3.16) into (3.15) allows us to conclude

$$\sum_{z \in Z_h} ||f||^2_{W_2^{k+\tau}(G(z, \bar{M}h))} \leqslant Q \sum_{i=1}^{2} \int_0^\infty \left(\frac{K(t, f_i)}{t^\tau} \right)^2 \frac{dt}{t} = Q ||f||^2_{W_2^{k+\tau}(S^{d-1})}. \qquad \square$$

4. A Sobolev extension theorem for the sphere

The aim of this section is to construct a continuous extension operator $E_{G(z,\theta)} : W_2^k(G(z,\theta)) \to W_2^k(S^{d-1})$, with the property that

$$(E_{G(z,\theta)}f)|_{G(z,\theta)} = f \quad \text{for all } f \in W_2^k(G(z,\theta)).$$

In view of (2.12) we start by extending the local functions $(\chi_i f)\circ\phi_i^{-1}|_{\Omega_i} \in W_2^k(\Omega_i)$ to $W_2^k(\mathbb{R}^{d-1})$ for $i \in \{1,2\}$.

Remark 4.1. If $\theta < C_{\mathcal{A}}/3$ then, by Lemma 2.3, we can restrict attention to the case where Ω_i is an open Euclidean ball, since otherwise $(\chi_i f)\circ\phi_i^{-1}|_{\Omega_i}$ is the zero function and thus has a trivial extension.

For the unit ball $B(0,1)$ and for $\varepsilon > 0$ sufficiently small, we can appeal to Theorem 1.1 for a continuous extension operator $E_{B(0,1)} : W_2^k(B(0,1)) \to W_2^k(\mathbb{R}^{d-1})$, where

$$\text{supp}(E_{B(0,1)}f) \subset B(0, 1+\varepsilon) \quad \text{for all } f \in W_2^k(B(0,1)).$$

To define an extension operator on $B(x,r)$ we use the coordinate transform

$$\sigma(y) = ry + x, \quad \text{for } r > 0 \quad \text{and } y \in \mathbb{R}^{d-1}, \tag{4.1}$$

and set

$$E_{B(x,r)}f(y) = (E_{B(0,1)}(f\circ\sigma))\circ\sigma^{-1}(y) \quad y \in \mathbb{R}^{d-1}. \tag{4.2}$$

In addition, we have that,

$$\text{supp}(E_{B(x,r)}f) \subset B(x, r(1+\varepsilon)) \quad \text{for all } f \in W_2^k(B(x,r)). \tag{4.3}$$

Remark 4.2. Let $z \in S^{d-1}$, $\theta < C_{\mathcal{A}}/3$, and assume that $\Omega_i = B(x_i, r_i) \subset B(0,1)$. We can use Lemma 2.3(ii) to choose any $\varepsilon < e_{\mathcal{A}}$ such that

$$\text{supp}(E_{B(x_i,r_i)}f) \subset B(x_i, r_i(1+\varepsilon)) \subset B(x_i, r_i+\varepsilon) \subset B(0,1).$$

Thus, by choosing a fixed positive $\varepsilon < e_{\mathcal{A}}$, which is independent of the centre z of the geodesic ball, we can ensure that $E_{B(x_i,r_i)}f$ is compactly supported in $B(0,1)$, for all $f \in W_2^k(B(x_i,r_i))$.

We are now in a position to prove the first extension theorem.

Theorem 4.3. *Let $z \in S^{d-1}$ and $\theta < C_{\mathcal{A}}/3$, then there exists an extension operator $E_{G(z,\theta)} : W_2^k(G(z,\theta)) \to W_2^k(S^{d-1})$ satisfying:*

1. $(E_{G(z,\theta)}f)|_{G(z,\theta)} = f$, *for every $f \in W_2^k(G(z,\theta))$,*
2. $\|E_{G(z,\theta)}f\|_{W_2^k(S^{d-1})} \leqslant \mathcal{K}\|f\|_{W_2^k(G(z,\theta))}$, *where $\mathcal{K}$ is independent of f and z.*

Proof. Let F_A denote the characteristic function of a set $A \subset S^{d-1}$. Let $f \in W_2^k(G(z,\theta))$ and $p \in S^{d-1}$, then we define a candidate extension operator $E_{G(z,\theta)} : W_2^k(G(z,\theta)) \to W_2^k(S^{d-1})$ by

$$E_{G(z,\theta)} f(p) = \sum_{i=1}^{2} E_{\Omega_i}((\chi_i f) \circ \phi_i^{-1}|_{\Omega_i})(\phi_i(p)) \cdot F_{U_i}(p). \tag{4.4}$$

Using Remark 4.1, we need only focus on the case where Ω_i is a Euclidean ball. Thus, we shall assume that z is located as in case $(3b)$ see Fig. 1. That is, $z \in U_1 \cap U_2$ and $\Omega_i = B(x_i, r_i)$, for $i \in \{1, 2\}$. In this case we have

$$E_{G(z,\theta)} f(p) = \sum_{i=1}^{2} E_{B(x_i,r_i)}((\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)})(\phi_i(p)), \tag{4.5}$$

where $E_{B(x_i,r_i)} : W_2^k(B(x_i, r_i)) \to W_2^k(\mathbb{R}^{d-1})$ is given by (4.2). In addition, we also choose $\varepsilon > 0$ as in Remark 4.1 to ensure that

$$\operatorname{supp}\{E_{B(x_i,r_i)}((\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)})\} \subset B(0, 1), \quad \text{for } i \in \{1, 2\}. \tag{4.6}$$

To prove part 1 we assume that $p \in G(z, \theta)$, that is, $\phi_i(p) \in B(x_i, r_i)$, for $i \in \{1, 2\}$. Then, by Theorem 1.1, we have

$$E_{B(x_i,r_i)}((\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)})(\phi_i(p)) = (\chi_i f) \circ \phi_i^{-1}(\phi_i(p)) = (\chi_i f)(p),$$

hence

$$E_{G(z,\theta)} f(p) = \sum_{i=1}^{2} (\chi_i f)(p) = f(p) \quad \text{as required.}$$

To prove part 2 we use (2.8) and consider

$$\begin{aligned} ||E_{G(z,\theta)} f||^2_{W_2^k(S^{d-1})} &= \sum_{j=1}^{2} ||\pi_j(\chi_j E_{G(z,\theta)} f)||^2_{W_2^k(\mathbb{R}^{d-1})} \\ &= \sum_{j=1}^{2} ||\pi_j(\chi_j) \pi_j(E_{G(z,\theta)} f)||^2_{W_2^k(\mathbb{R}^{d-1})}. \end{aligned}$$

We note that $\pi_j(\chi_j) \in C_0^\infty(\mathbb{R}^{d-1})$ and so there exists a constant $\mathcal{K}_\chi$ depending only on $\mathcal{A}$ and the partition of unity $\{\chi_j\}_{j=1}^2$ such that

$$\begin{aligned} &||E_{G(z,\theta)} f||^2_{W_2^k(S^{d-1})} \\ &\quad \leqslant \mathcal{K}_\chi \sum_{j=1}^{2} ||\pi_j(E_{G(z,\theta)} f)||^2_{W_2^k(\mathbb{R}^{d-1})} \end{aligned}$$

$$= \mathcal{K}_\chi \sum_{j=1}^{2} \left\| \sum_{i=1}^{2} E_{B(x_i,r_i)}((\chi_i f)\circ\phi_i^{-1}|_{B(x_i,r_i)})(\phi_i\circ\phi_j^{-1}) \right\|^2_{W_2^k(\mathbb{R}^{d-1})} \quad \text{by (4.5)}$$

$$= \mathcal{K}_\chi \sum_{j=1}^{2} \left\| \sum_{i=1}^{2} E_{B(x_i,r_i)}((\chi_i f)\circ\phi_i^{-1}|_{B(x_i,r_i)})(\phi_i\circ\phi_j^{-1}) \right\|^2_{W_2^k(B(0,1))} \quad \text{by (4.6)}$$

$$\leqslant 2\mathcal{K}_\chi \sum_{j=1}^{2} \sum_{i=1}^{2} ||E_{B(x_i,r_i)}((\chi_i f)\circ\phi_i^{-1}|_{B(x_i,r_i)})(\phi_i\circ\phi_j^{-1})||^2_{W_2^k(B(0,1))}.$$

Since $\mathcal{A} = \{U_i, \phi_i\}_{i=1}^2$ is an atlas for S^{d-1}, the coordinate changes, $\phi_i\circ\phi_j^{-1}: \phi_j(U_i\cap U_j)\to\phi_i(U_i\cap U_j)$, for $i\neq j\in\{1,2\}$, are infinitely differentiable. Therefore there exists a constant $\mathcal{K}_\mathcal{A}$, depending only on $\mathcal{A}$, such that

$$||E_{G(z,\theta)}f||^2_{W_2^k(S^{d-1})} \leqslant 2\mathcal{K}_\chi\mathcal{K}_\mathcal{A} \sum_{j=1}^{2} \sum_{i=1}^{2} ||E_{B(x_i,r_i)}((\chi_i f)\circ\phi_i^{-1}|_{B(x_i,r_i)})||^2_{W_2^k(B(0,1))}$$

$$\leqslant 4\mathcal{K}_\chi\mathcal{K}_\mathcal{A} \sum_{i=1}^{2} ||E_{B(x_i,r_i)}((\chi_i f)\circ\phi_i^{-1}|_{B(x_i,r_i)})||^2_{W_2^k(\mathbb{R}^{d-1})}. \tag{4.7}$$

The function $(\chi_i f)\circ\phi_i^{-1}|_{B(x_i,r_i)}$ belongs to $W_2^k(B(x_i,r_i))$, for $i\in\{1,2\}$. Thus, we can appeal to Theorem 1.1 to deduce the existence of a constant $\mathcal{C}_{\text{ext}}$, independent of $(\chi_i f)\circ\phi_i^{-1}|_{B(x_i,r_i)}$ (and therefore of f), such that

$$||E_{G(z,\theta)}f||^2_{W_2^k(S^{d-1})} \leqslant 4\mathcal{K}_\chi\mathcal{K}_\mathcal{A}\mathcal{C}^2_{\text{ext}} \sum_{i=1}^{2} ||(\chi_i f)\circ\phi_i^{-1}|_{B(x_i,r_i)}||^2_{W_2^k(B(x_i,r_i))}.$$

Taking square roots gives

$$||E_{G(z,\theta)}f||_{W_2^k(S^{d-1})} \leqslant \mathcal{K}||f||_{W_2^k(G(z,\theta))}, \quad \text{where } \mathcal{K} = \mathcal{C}_{\text{ext}}\sqrt{4\mathcal{K}_\chi\mathcal{K}_\mathcal{A}}, \tag{4.8}$$

and this completes the proof for z as in case (3b); the proof for the other cases follows in a similar, but simpler, fashion. □

We now turn to the extension constant $\mathcal{K}$ (4.8) of the operator $E_{G(z,\theta)}$. In particular, we shall investigate its dependence upon the geodesic radius θ. By inspection, it is clear that the factors $\mathcal{K}_\chi$ and $\mathcal{K}_\mathcal{A}$ are both independent of θ. However, the factor $\mathcal{C}_{\text{ext}}$ may depend upon the radii r_1 or r_2 and these are both related to θ by (2.16). Indeed, this dependence is established as follows.

Lemma 4.4. *Let $k > \frac{d-1}{2}$ be a non-negative integer and let $E_{B(x,r)}: W_2^k(B(x,r)) \to W_2^k(\mathbb{R}^{d-1})$ be a linear extension operator such that, for all $f \in W_2^k(B(x,r))$,*

$$||E_{B(x,r)}f||_{W_2^k(\mathbb{R}^{d-1})} \leqslant \mathcal{C}_{\text{ext}}||f||_{W_2^k(B(x,r))}. \tag{4.9}$$

Then the extension constant $\mathcal{C}_{\text{ext}} > 0$ *is necessarily dependent upon the radius r of the ball.*

Proof. Since $k > \frac{d-1}{2}$ the Sobolev embedding theorem [2] tells us that $W_2^k(\mathbb{R}^{d-1})$ is a space of continuous functions. Hence, there exists a constant $c > 0$ such that

$$c \cdot |f(y)| \leqslant ||f||_{W_2^k(\mathbb{R}^{d-1})}, \quad \text{for all } f \in W_2^k(\mathbb{R}^{d-1}) \quad \text{and } y \in \mathbb{R}^{d-1}. \tag{4.10}$$

Let E denote a continuous linear extension operator and choose $f \in W_2^k(B(x,r))$ to be $f = 1$. A combination of (4.9) and (4.10) yields

$$c \leqslant ||E_{B(x,r)} f||_{W_2^k(\mathbb{R}^{d-1})} \leqslant \mathcal{C}_{\text{ext}} ||f||_{W_2^k(B(x,r))}. \tag{4.11}$$

Now since $||f||_{W_2^k(B(x,r))} \to 0$ as $r \to 0$ we can deduce, from (4.11), that the constant $\mathcal{C}_{\text{ext}}$ must grow to ∞ as $r \to 0$. Hence $\mathcal{C}_{\text{ext}}$ is necessarily dependent on the radius of the ball. □

5. A restricted Sobolev extension theorem

We will now show that if $E_{G(z,\theta)}$ is restricted to a certain Sobolev subspace, then it has an extension constant $\widetilde{\mathcal{K}}$ which is independent of both z and θ. We begin, again, by investigating the analogous problem in the Euclidean setting.

5.1. The Euclidean case

Let $k > \frac{d-1}{2}$, $x^* \in \mathbb{R}^{d-1}$, and consider the extension of continuous functions from $W_2^k(B(x^*, r))$ to $W_2^k(\mathbb{R}^{d-1})$. To simplify matters, we shall assume that $B(x^*, r) \subset B(0, 1)$ and focus only on the operator

$$E_{B(x^*,r)} f(y) = (E_{B(0,1)}(f \circ \sigma)) \circ \sigma^{-1}(y) \quad y \in \mathbb{R}^{d-1},$$

where $E_{B(0,1)}$ is supplied by Theorem 1.1, and where σ is given by (4.1). One advantage of this construction is that, for any $f \in W_2^k(B(x^*, r))$, we can consider the translated and scaled function $f \circ \sigma \in W_2^k(B(0, 1))$. In particular, we have the following useful change of variables result.

Lemma 5.1. *Let* $\alpha \in \mathbb{N}_0^d$ *be a multi-index with* $|\alpha| \leqslant k$. *Then, for any* $f \in W_2^k(B(x^*, r))$, *we have*

$$||D^\alpha f||^2_{L_2(B(x^*,r))} = r^{(d-1)-2|\alpha|} ||D^\alpha (f \circ \sigma)||^2_{L_2(B(0,1))}, \tag{5.1}$$

and similarly, for any $f \in W_2^k(\mathbb{R}^{d-1})$, *we have*

$$||D^\alpha f||^2_{L_2(\mathbb{R}^{d-1})} = r^{(d-1)-2|\alpha|} ||D^\alpha (f \circ \sigma)||^2_{L_2(\mathbb{R}^{d-1})}. \tag{5.2}$$

Proof. See [4, Lemma 4.9]. □

Let $X = \{x_i\}_{i=1}^N$ denote a set of distinct points in $B(x^*, r)$. We measure the density of X in $B(x^*, r)$ by assigning the local Euclidean mesh norm

$$\rho := \rho(X, B(x^*, r)) = \sup_{y \in B(x^*,r)} \min\{||x - y|| : x \in X\}. \tag{5.3}$$

Remark 5.2. Assuming that the set $X = \{x_i\}_{i=1}^N$ has mesh-norm ρ in $B(x^*, r)$, then its inverse image $\sigma^{-1}(X)$ has mesh norm ρ/r in $B(0, 1)$.

We have shown, in Lemma 4.4, that the extension constant $\mathcal{C}_{\text{ext}}$ of $E_{B(x^*,r)}$ necessarily depends on r. To overcome this, we consider the following subspace:

$$\widetilde{W}_2^k(B(x^*, r)) = \{f \in W_2^k(B(x^*, r)) \colon f(x) = 0,\ x \in X\}, \tag{5.4}$$

where X is a set of distinct points in $B(x^*, r)$. We aim to show that if the local Euclidean mesh norm ρ of X is small enough then the restriction of $E_{B(x^*,r)}$ to $\widetilde{W}_2^k(B(x^*, r))$ has an extension constant $\widetilde{\mathcal{C}}_{\text{ext}}$ independent of both x^* and r.

We begin by providing some background material. First of all, for a fixed integer $k > 1$ we let $\Pi_{k-1}(\mathbb{R}^d)$ denote the space of all d-variate polynomials of degree at most $k - 1$. The dimension of this space is

$$M_{k-1} = \dim \Pi_{k-1}(\mathbb{R}^d) = \binom{k + d - 1}{d}. \tag{5.5}$$

Let $X_k = \{x_i\}_{i=1}^{M_{k-1}}$ denote a set of distinct points in $\mathbb{R}^d$, we say that the set X_k is Π_{k-1}-*unisolvent* if the only $p \in \Pi_{k-1}(\mathbb{R}^d)$ to vanish on X_k is the zero polynomial. Furthermore, if $X_k \subset B(x^*, r)$ then we say that the M_{k-1}-tuple

$$(x_1, \ldots, x_{M_{k-1}}) \in \underbrace{B(x^*, r) \times \cdots \times B(x^*, r)}_{M_{k-1}\ \text{times}} = B(x^*, r)^{M_{k-1}}$$

is Π_{k-1}-*unisolvent in* $B(x^*, r)$. Let $\mathcal{U}$ denote the set of all M_{k-1}-tuples,

$$(x_1, \ldots, x_{M_{k-1}}) \in \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{M_{k-1}\ \text{times}} = (\mathbb{R}^d)^{M_{k-1}}.$$

that are Π_{k-1}-*unisolvent in* $\mathbb{R}^{d-1}$, that is,

$$\mathcal{U} = \{(x_1, \ldots, x_{M_{k-1}}) : \{x_1, \ldots, x_{M_{k-1}}\} \text{ is } \Pi_{k-1}\text{-unisolvent}\}.$$

We note that $\mathcal{U}$ is open (its complement is the set of solutions of algebraic equations). Thus, for a given $(x_1, \ldots, x_{M_{k-1}}) \in \mathcal{U}$, there exists $\delta > 0$, such that

$$\overline{B(x_1, \delta)} \times \cdots \times \overline{B(x_{M_{k-1}}, \delta)} \subset \mathcal{U}. \tag{5.6}$$

Thus a Π_{k-1}-unisolvent set of points remains Π_{k-1}-unisolvent after a small perturbation. This observation is summarised in the following result.

Lemma 5.3. *Let*

$$(\hat{x}_1, \ldots, \hat{x}_{M_{k-1}}) \in \underbrace{B(0,1) \times \cdots \times B(0,1)}_{M_{k-1} \text{ times}} \in B(0,1)^{M_{k-1}}$$

be Π_{k-1}*-unisolvent in* $B(0,1)$. *Then there exists* $\delta_{k-1} > 0$ *such that* $\overline{B(\hat{x}_i, \delta_{k-1})} \subset B(0,1)$, *for* $1 \leqslant i \leqslant M_{k-1}$, *and so that each element of*

$$\mathcal{W} = \overline{B(\hat{x}_1, \delta_{k-1})} \times \cdots \times \overline{B(\hat{x}_{M_{k-1}}, \delta_{k-1})},$$

is Π_{k-1}*-unisolvent in* $B(0,1)$.

The next result shows how a sufficiently dense set of points in $B(x^*, r)$ can be mapped, under σ^{-1}, to a set in $B(0,1)$ containing a Π_{k-1}-unisolvent subset.

Proposition 5.4. *Let* $\mathcal{W}$, M_{k-1} *and* δ_{k-1} *be as in Lemma* 5.3. *Let* $X = \{x_i\}_{i=1}^N$ *denote a set of* $N \geqslant M_{k-1}$ *distinct points in* $B(x^*, r)$, *with local mesh norm* ρ (5.3). *If* $\rho/r < \delta_{k-1}$, *then there exists* M_{k-1} *distinct points*,

$$\{w_1, \ldots, w_{M_{k-1}}\} \subset \sigma^{-1}(X) = \{\sigma^{-1}(x_1), \ldots, \sigma^{-1}(x_N)\},$$

such that $(w_1, \ldots, w_{M_{k-1}}) \in \mathcal{W}$.

Proof. By Remark 5.2, the mesh norm of $\sigma^{-1}(X)$ in $B(0,1)$ is ρ/r. Thus, for each of the points $\hat{x}_i \in B(0,1)$ from Lemma 5.3, we have that

$$\min_{1 \leqslant j \leqslant N} ||\sigma^{-1}(x_j) - \hat{x}_i|| < \rho/r < \delta_{k-1}, \quad \text{for } 1 \leqslant i \leqslant M_{k-1}.$$

In other words, each $\overline{B(\hat{x}_i, \delta_{k-1})}$ contains at least one element, w_i say, from $\sigma^{-1}(X)$, and hence the result follows from Lemma 5.3. □

To complete our background work, we state a specialisation of an important result due to Duchon [3].

Lemma 5.5 (Duchon). *Let* $k > \frac{d-1}{2}$ *be a positive integer, let* $B(0,1) \subset \mathbb{R}^{d-1}$ *and let* $\mathcal{W}$ *be as in Lemma* 5.3. *Then, for each* $i \leqslant k$, *there exists a constant* $C_{\mathcal{W}}(i)$ *depending on* $B(0,1)$, $\mathcal{W}$, k *and* i *such that*

$$\sum_{|\beta|=i} ||D^\beta f||^2_{L_2(B(0,1))} \leqslant C_{\mathcal{W}}(i) \cdot \sum_{|\beta|=k} ||D^\beta f||^2_{L_2(B(0,1))} \tag{5.7}$$

for all $f \in W_2^k(B(0,1))$ *such that* $f(w_j) = 0$, $(1 \leqslant j \leqslant M_{k-1})$, *for some* $(w_1, \ldots, w_{M_{k-1}}) \in \mathcal{W}$.

Proof. See Section 2 of [3]. □

We are now able to prove the following restricted extension theorem.

Theorem 5.6. *Let $k > \frac{d-1}{2}$ be a positive integer, and let $\mathcal{W}$, M_{k-1} and δ_{k-1} be as in Lemma* 5.3. *Let X denote a set of distinct points in $B(x^*, r) \subset \mathbb{R}^{d-1}$ whose mesh norm ρ* (5.3) *satisfies*

$$\rho/r < \delta_{k-1}.$$

Let $E_{B(x^*,r)} : W_2^k(B(x^*, r)) \to W_2^k(\mathbb{R}^{d-1})$ be the extension operator given by (4.2). *Then there exists a constant $\widetilde{\mathcal{C}}_{\text{ext}}$ independent of x^* and r such that*

$$||E_{B(x^*,r)} f||_{W_2^k(\mathbb{R}^{d-1})} \leqslant \widetilde{\mathcal{C}}_{\text{ext}} ||f||_{W_2^k(B(x^*,r))} \quad \text{for all } f \in \widetilde{W}_2^k(B(x^*, r)). \tag{5.8}$$

Proof. Let $f \in \widetilde{W}_2^k(B(x^*, r))$, then, applying (1.6), (5.2), and (4.2), respectively, we have

$$||E_{B(x^*,r)} f||^2_{W_2^k(\mathbb{R}^{d-1})} = \sum_{0 \leqslant |\alpha| \leqslant k} r^{(d-1)-2|\alpha|} ||D^\alpha (E_{B(0,1)}(f \circ \sigma))||^2_{L_2(\mathbb{R}^{d-1})}.$$

Furthermore, since $r \in (0, 1)$ we can deduce that

$$||E_{B(x^*,r)} f||^2_{W_2^k(\mathbb{R}^{d-1})} \leqslant r^{(d-1)-2k} \cdot ||E_{B(0,1)}(f \circ \sigma)||^2_{W_2^k(\mathbb{R}^{d-1})}.$$

By Theorem 1.1 there exists a constant $C_{B(0,1)}$, independent of x and r, such that

$$||E_{B(x^*,r)} f||^2_{W_2^k(\mathbb{R}^{d-1})} \leqslant C_{B(0,1)} \cdot r^{(d-1)-2k} \cdot ||f \circ \sigma||^2_{W_2^k(B(0,1))}. \tag{5.9}$$

We observe that:

(i) The assumption $\rho/r < \delta_{k-1}$ implies, by Lemma 5.4, that there exists a set of distinct points $\{w_1, \ldots, w_{M_{k-1}}\} \subset \sigma^{-1}(X)$, such that $(w_1, \ldots, w_{M_{k-1}}) \in \mathcal{W} \subset B(0, 1)^{M_{k-1}}$.

(ii) The function $f \circ \sigma \in W_2^k(B(0, 1))$ vanishes at each point in $\sigma^{-1}(X)$, and hence at each w_i for $i = 1, \ldots, M_{k-1}$.

Together, (i) and (ii) allow us to apply Lemma 5.5. Thus, setting $A^{\mathcal{W}} = \max\{C_{\mathcal{W}}(i) : i = 0, \ldots, k\}$, we can employ (1.6), (5.7) and (5.1) to deduce that

$$\begin{aligned} ||f \circ \sigma||^2_{W_2^k(B(0,1))} &\leqslant A^{\mathcal{W}} r^{-(d-1)+2k} \sum_{|\alpha|=k} ||D^\alpha f||^2_{L_2(B(x^*,r))} \\ &\leqslant A^{\mathcal{W}} r^{-(d-1)+2k} \sum_{0 \leqslant |\alpha| \leqslant k} ||D^\alpha f||^2_{L_2(B(x^*,r))}. \end{aligned}$$

That is, we have

$$||f \circ \sigma||^2_{W_2^k(B(0,1))} \leqslant A^{\mathcal{W}} \cdot r^{-(d-1)+2k} ||f||^2_{W_2^k(B(x^*,r))}. \tag{5.10}$$

Substituting (5.10) into (5.9) and taking square roots yields,

$$||E_{B(x^*,r)} f||_{W_2^k(\mathbb{R}^{d-1})} \leqslant \sqrt{C_{B(0,1)} \cdot A^{\mathcal{W}}} \cdot ||f||_{W_2^k(B(x^*,r))}.$$

We note that the constant $A^{\mathcal{W}}$ is independent of x^* and r, and hence, setting $\widetilde{\mathcal{C}}_{\text{ext}} = \sqrt{C_{B(0,1)} \cdot A^{\mathcal{W}}}$ completes the proof. □

5.2. The spherical case

In Theorem 4.3, we constructed an extension operator $E_{G(z,\theta)}: W_2^k(G(z,\theta)) \to W_2^k(S^{d-1})$. Following the proof of this theorem to Eq. (4.7), we have

$$||E_{G(z,\theta)}f||^2_{W_2^k(S^{d-1})} \leqslant 4\mathcal{K}_\chi \mathcal{K}_\mathcal{A} \sum_{i=1}^{2} ||E_{B(x_i,r_i)}f_i||^2_{W_2^k(\mathbb{R}^{d-1})}, \tag{5.11}$$

where

$$f_i = (\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)} \quad \text{for } i \in \{1,2\},$$

and where the constants $\mathcal{K}_\chi$ and $\mathcal{K}_\mathcal{A}$ are independent of z and θ.

Observation 5.7. *If, for a given $f \in W_2^k(G(z,\theta))$, the projected functions $f_i \in W_2^k(B(x_i,r_i))$, $i \in \{1,2\}$, vanish on a sufficiently dense set of points in $B(x_i,r_i)$, then we could use a combination of Theorem* 5.6 *and Lemma* 2.4 *to conclude that $E_{G(z,\theta)}$ extends f independently of z and θ.*

The above observation provides us with the strategy to prove the first restricted extension theorem for the sphere. To begin with we let $\Xi = \{\xi_i\}_{i=1}^N$ denote a set distinct points on S^{d-1} whose density is measured using the mesh norm

$$h := h(\Xi, S^{d-1}) := \sup_{\eta \in S^{d-1}} \min\{g(\eta,\xi_i) = \cos^{-1}(\eta^T \xi_i) : \xi_i \in \Xi\}. \tag{5.12}$$

For a positive integer $k > \frac{d-1}{2}$, we consider the following subspace

$$\widetilde{W}_2^k(G(z,\theta)) = \{f \in W_2^k(G(z,\theta)) : f(\xi_i) = 0, \quad \xi_i \in \Xi\}. \tag{5.13}$$

Suppose that $f \in \widetilde{W}_2^k(G(z,\theta))$. Then, for $i \in \{1,2\}$, the projected functions $f_i = (\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)} \in W_2^k(B(x_i,r_i))$ vanish on the transformed set of points given by

$$X_\theta^{(i)} = \{\phi_i(\xi) : \xi \in \Xi \cap G(z,\theta)\} \subset B(x_i,r_i). \tag{5.14}$$

In summary, we conclude that

$$f \in \widetilde{W}_2^k(G(z,\theta)) \Rightarrow f_i = (\chi_i f) \circ \phi_i^{-1}|_{B(x_i,r_i)} \in \widetilde{W}_2^k(B(x_i,r_i)), \quad \text{for } i \in \{1,2\},$$

where

$$\widetilde{W}_2^k(B(x_i,r_i)) = \{f \in W_2^k(B(x_i,r_i)) : f(x) = 0, \text{ for } x \in X_\theta^{(i)}\}. \tag{5.15}$$

For $i \in \{1,2\}$, we measure the density of $X_\theta^{(i)}$, which we assume to be non-empty, by assigning the local Euclidean mesh norm

$$\rho_i = \sup_{x \in B(x_i,r_i)} \min\{||x - \phi_i(\xi)|| : \xi \in \Xi \cap G(z,\theta)\}. \tag{5.16}$$

Remark 5.8. Let $k > \frac{d-1}{2}$ be a positive integer and let δ_{k-1} be as in Lemma 5.3. Let $f \in \widetilde{W}_2^k(G(z,\theta))$ and assume that the local Euclidean mesh norms (5.16) satisfy

$$\frac{\rho_i}{r_i} < \delta_{k-1}, \quad \text{for } i \in \{1,2\}. \tag{5.17}$$

Then, using Theorem 5.6, there exist a constant $\widetilde{\mathcal{C}}_{\text{ext}}$, independent of x_i and r_i, $i \in \{1,2\}$, such that

$$||E_{G(z,\theta)}f||^2_{W_2^k(S^{d-1})} \leqslant 4\mathcal{K}_\chi \mathcal{K}_\mathcal{A} \widetilde{\mathcal{C}}^2_{\text{ext}} \sum_{i=1}^{2} ||f_i||^2_{W_2^k(B(x_i,r_i))} = \widetilde{\mathcal{K}}^2 ||f||^2_{W_2^k(G(z,\theta))}.$$

where $\widetilde{\mathcal{K}} = \sqrt{4\mathcal{K}_\chi \mathcal{K}_\mathcal{A}} \widetilde{\mathcal{C}}_{\text{ext}}$ is independent of z and θ.

Remark 5.8 provides the route to the initial restricted extension theorem for the sphere. In order to make this rigorous we require the geometrical arguments which relate the density of Ξ in S^{d-1} to the densities of the $X_\theta^{(i)}$ in $B(x_i, r_i)$, for $i \in \{1,2\}$.

Geometrical arguments. Let $\Xi = \{\xi_i\}_{i=1}^N$ denote a set of distinct points on S^{d-1} whose density is measured by the mesh norm h given by (5.12). To measure the density of Ξ locally, on some $G(z,\theta)$ say, we assign a local mesh norm by

$$h_L = \sup_{\eta \in G(z,\theta)} \min\{g(\eta,\xi) : \xi \in \Xi \cap G(z,\theta)\}. \tag{5.18}$$

The following result provides a relationship between h and h_L.

Lemma 5.9. *Let Ξ be a set of points on S^{d-1} with mesh norm $h \in (0, \pi/6)$. Let $z \in S^{d-1}$, $\theta \geqslant 3h$ and let h_L denote the local mesh norm of $\Xi \cap G(z,\theta)$. Then*

$$h_L \leqslant 4h. \tag{5.19}$$

Proof. Let $\eta \in G(z,\theta)$ and let ξ be a closest point to η from Ξ. Then, by (5.12), we have $g(\eta,\xi) \leqslant h$. We prove the lemma by splitting into two cases based on the position of ξ.

(a) If $\xi \in G(z,\theta)$, then $\min\{g(\eta,\xi) : \xi \in \Xi \cap G(z,\theta)\} \leqslant h < 4h$.

(b) If $\xi \notin G(z,\theta)$, then $g(\xi,z) \geqslant \theta \geqslant 3h$. Thus, there exists a point $\eta' \in G(z,\theta)$, lying on the intersection between the boundary of $G(\xi, 2h)$ and the geodesic arc connecting z and ξ, (see Fig. 4). That is, η' satisfies

$$g(z,\xi) = g(z,\eta') + g(\eta',\xi) = g(z,\eta') + 2h.$$

Furthermore, there must exist a $\xi' \in \Xi$, such that $g(\eta',\xi') \leqslant h$. The triangle inequality allows us to deduce

$$\begin{aligned} g(z,\xi') &\leqslant g(z,\eta') + g(\eta',\xi') = g(z,\xi) - 2h + g(\eta',\xi') \\ &\leqslant g(z,\eta) + g(\eta,\xi) - h < \theta + h - h = \theta. \end{aligned}$$

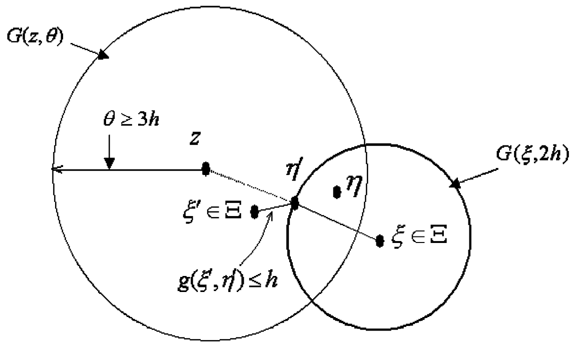

Fig. 4. Illustration of Lemma 5.10.

Thus, $\xi' \in G(z,\theta)$, and this implies

$$g(\eta,\xi') \leqslant g(\eta,\xi) + g(\xi,\eta') + g(\eta',\xi') \leqslant h + 2h + h = 4h.$$

Hence $\min\{g(\eta,\xi) : \xi \in \Xi \cap G(z,\theta)\} \leqslant 4h$. These arguments hold for any $\eta \in G(z,\theta)$ and so, by (5.18), the proof is complete. $\square$

The next result shows how the geodesic mesh norm of Ξ relates to the local Euclidean mesh norms of the $X_\theta^{(i)}$, for $i \in \{1,2\}$.

Lemma 5.10. *Let Ξ be a set of points on S^{d-1} with mesh norm $h \in (0,\pi/6)$. Let $z \in S^{d-1}$, $\theta \geqslant 3h$ and assume that $\overline{G(z,\theta)} \subset U_i$, $i \in \{1,2\}$. Let ρ_i denote the Euclidean mesh norm of $X_\theta^{(i)} = \phi_i(\Xi \cap G(z,\theta))$ given by Eq.* (5.16), *then*

$$\rho_i \leqslant 4C_0 h, \quad \textit{for } i \in \{1,2\}, \tag{5.20}$$

where C_0 is as in Lemma 2.4.

Proof. Let $x \in B(x_i,r_i)$, then $\eta = \phi_i^{-1}(x) \in G(z,\theta)$. By Lemma 5.9, there exists a point $\xi \in \Xi$ such that $g(\eta,\xi) \leqslant 4h$. We note that $4h \leqslant 2\pi/3$ and so we can use Lemma 2.4 to deduce that

$$\min_{\xi \in \Xi \cap G(z,\theta)} ||\phi_i(\eta) - \phi_i(\xi)|| = \min_{\xi \in \Xi \cap G(z,\theta)} ||x - \phi_i(\xi)|| \leqslant 4C_0 h.$$

This result holds for all $x \in B(x_i,r_i)$ and so proves the lemma. $\square$

The extension theorem for integer order spaces. Let Ξ denote the usual set of distinct points in S^{d-1} with mesh norm $h \in (0,\pi/6)$, given by (5.12). Assume that $\theta < C_{\mathcal{A}}/3$, and let $R > 0$, be such that $\theta = Rh$. Using Lemma 5.10, we can deduce that if $R \geqslant 3$, then $\rho_i \leqslant 4C_0 \cdot h$, for $i \in \{1,2\}$. Furthermore, using Lemma 2.4, there exists a constant $c_0 > 0$, such that

$$\frac{\rho_i}{r_i} \leqslant \frac{4C_0 \cdot h}{r_i} \leqslant \frac{4C_0 \cdot h}{c_0 R h} = \frac{4C_0}{c_0 R}, \quad \text{for } i \in \{1,2\}. \tag{5.21}$$

Thus, if $R \geqslant 3$ and $R > 4C_0/c_0\delta_{k-1}$ then condition (5.17) holds. In view of this, by setting

$$\mathcal{R}_0 = \max\left\{3, \frac{4C_0}{c_0\delta_{k-1}}\right\}, \tag{5.22}$$

we are able to formulate the following extension theorem.

Theorem 5.11. *Let $k > \frac{d-1}{2}$ be a positive integer and let Ξ denote a set of distinct points on S^{d-1} whose mesh norm h satisfies*

(i) $h \in (0, \pi/6)$, (ii) $\mathcal{R}_0 h < C_{\mathcal{A}}/3$,

where $\mathcal{R}_0$ is given by (5.22). *Let $z \in S^{d-1}$, $\theta \in (\mathcal{R}_0 h, C_{\mathcal{A}}/3)$, and let $E_{G(z,\theta)}$ denote the continuous extension operator given by* (4.5). *Then there exists a constant $\widetilde{\mathcal{K}} > 0$, independent of z and θ, such that*

$$||E_{G(z,\theta)} f||_{W_2^k(S^{d-1})} \leqslant \widetilde{\mathcal{K}} ||f||_{W_2^k(G(z,\theta))} \quad f \in \widetilde{W}_2^k(G(z,\theta)). \tag{5.23}$$

Proof. The conditions of the theorem guarantee that the local mesh norms $\{\rho_i\}_{i=1}^2$ of the transformed point sets $\{X_\theta^{(i)}\}_{i=1}^2$ satisfy (5.17). The theorem then follows from the arguments set out in Remark 5.8. □

The extension theorem for fractional order spaces. We motivate this section by recalling some standard Banach space theory.

Definition 5.12. A closed linear subspace $\widetilde{A}$ of a Banach space A is said to be a *complemented subspace* of A if and only if there exists a continuous projection $\mathcal{P}$ on A with $\mathcal{P}(A) = \widetilde{A}$.

Let (A_0, A_1) be an interpolation pair and let $\mathcal{P}$ denote a projection operator acting upon both A_0 and A_1. Since a complemented subspace of a Banach space is closed, we can consider the spaces $\widetilde{A}_0 = \mathcal{P}(A_0)$ and $\widetilde{A}_1 = \mathcal{P}(A_1)$ as Banach space in their own right, with the inherited norms $||\cdot||_{A_0}$ and $||\cdot||_{A_1}$, respectively. Furthermore, $(\widetilde{A}_0, \widetilde{A}_1)$ is itself a valid interpolation pair. The following result is due to Triebel, see [10]; Section 1.17.

Theorem 5.13. *Let $\{A_0, A_1\}$ and $\{\widetilde{A}_0, \widetilde{A}_1\}$ denote two interpolation pairs, where $\widetilde{A}_0$ and $\widetilde{A}_1$ are the complemented subspaces of A_0 and A_1 respectively, with common projection operator $\mathcal{P}$. Then, for $\tau \in (0,1)$, we have*

$$\widetilde{A}_\tau = (\widetilde{A}_0, \widetilde{A}_1)_\tau = \mathcal{P}(A_0, A_1)_\tau = \mathcal{P}(A_\tau). \tag{5.24}$$

That is, $\widetilde{A}_\tau$ is the complemented subspace of A_τ with the same projection $\mathcal{P}$.

We illustrate this material with the following Sobolev space result.

Proposition 5.14. *Let $k > \frac{d-1}{2}$, be a positive integer. Let $X = \{x_i\}_{i=1}^N$ denote a set of distinct points in $B(x^*, r) \subset \mathbb{R}^{d-1}$. The familiar Sobolev subspace*

$$\widetilde{W}_2^k(B(x^*, r)) = \{f \in W_2^k(B(x^*, r)) : f(x) = 0, \ x \in X\},$$

is a complemented subspace of $W_2^k(B(x^*, r))$.

Proof. We can choose a set of N linearly independent cardinal functions $\hat{\gamma}_i \in W_2^k(B(x^*, r))$ with the following properties:

(i) $\hat{\gamma}_i(x_i) = 1$, for $i = 1, \ldots, N$,
(ii) $\hat{\gamma}_i$ has compact support $K_i \subset B(x^*, r)$ and $K_i \cap K_j = \emptyset$ whenever $i \neq j$.

Then, since $k > \frac{d-1}{2}$, we can define a projection operator on $W_2^k(B(x^*, r))$ by

$$\mathcal{Q}_X : f \mapsto \sum_{i=1}^N f(x_i)\hat{\gamma}_i. \tag{5.25}$$

We note that the null space of $\mathcal{Q}_X$ is precisely $\widetilde{W}_2^k(B(x^*, r))$. Thus, setting $\mathcal{P}_X := \mathcal{I} - \mathcal{Q}_X$, where $\mathcal{I}$ denotes the identity, completes the proof. □

For $k > \frac{d-1}{2}$, we can deduce that $(\widetilde{W}_2^k(B(x^*, r)), \widetilde{W}_2^{k+1}(B(x^*, r)))$ is a valid interpolation pair. Thus, for $\tau \in (0, 1)$, we can define its interpolation space

$$\widetilde{W}_2^{k+\tau}(B(x^*, r)) = (\widetilde{W}_2^k(B(x^*, r)), \widetilde{W}_2^{k+1}(B(x^*, r)))_\tau. \tag{5.26}$$

In particular, as a corollary of Theorem 5.13, we have the following result.

Theorem 5.15. *For $k > \frac{d-1}{2}$, and $\tau \in (0, 1)$, we have that*

$$\widetilde{W}_2^{k+\tau}(B(x^*, r)) = \{f \in W_2^{k+\tau}(B(x^*, r)) : f(x) = 0, \quad \text{for } x \in X\}.$$

The final aim of this paper is to prove a fractional version of Theorem 5.11. We begin by considering the Euclidean setting where we have the following intermediate results.

Proposition 5.16. *Let $\tau \in (0, 1)$ and $k > \frac{d-1}{2}$ be a positive integer. Let $E_{B(x^*,r)}$ be the extension operator given by* (4.2), *which maps $W_2^{k+i}(B(x^*, r))$ to $W_2^{k+i}(\mathbb{R}^{d-1})$, for $i = 0, 1$. Then*

(i) $E_{B(x^*,r)} : W_2^{k+\tau}(B(x^*, r)) \to W_2^{k+\tau}(\mathbb{R}^{d-1})$,
(ii) $(E_{B(x^*,r)} f)|_{B(x,r)} = f$, *for all $f \in W_2^{k+\tau}(B(x^*, r))$,*
(iii) $||Ef||_{W_2^{k+\tau}(\mathbb{R}^{d-1})} \leqslant \mathcal{C}_{\text{ext}}^{(\tau)} ||f||_{W_2^{k+\tau}(B(x^*,r))}$, *where $\mathcal{C}_{\text{ext}}^{(\tau)}$ is independent of f.*

Proof. Parts (i) and (iii) are true by the operator interpolation property. Also, property (ii) holds for all of the integer order spaces (cf. Theorem 1.1), it also holds for the fractional spaces since $W_2^{k+\tau}(B(x^*,r)) \subset W_2^k(B(x^*,r))$. $\square$

Proposition 5.17. *Let $\tau \in (0,1)$ and $k > \frac{d-1}{2}$ be a positive integer. Let $X = \{x_i\}_{i=1}^N$ denote a set of distinct points in $B(x^*,r) \subset B(0,1)$ whose mesh norm ρ satisfies*

$$\frac{\rho}{r} < \delta_k, \tag{5.27}$$

where δ_k is as in Lemma 5.3. Then there exists a constant $\widetilde{\mathcal{C}}_{\text{ext}}^{(\tau)}$ independent of x^* and r such that

$$\|E_{B(x^*,r)}f\|_{W_2^{k+\tau}(\mathbb{R}^{d-1})} \leqslant \widetilde{\mathcal{C}}_{\text{ext}}^{(\tau)} \|f\|_{W_2^{k+\tau}(B(x^*,r))}, \quad f \in \widetilde{W}_2^{k+\tau}(B(x^*,r)). \tag{5.28}$$

Proof. We recall, from Theorem 5.15, that

$$\widetilde{W}_2^{k+\tau}(B(x^*,r)) := \{f \in W_2^{k+\tau}(B(x^*,r)) : f(x) = 0 \quad x \in X\}.$$

The assumption $\rho/r < \delta_k$, allows us to deduce, from Theorem 5.6, that

$$\|E_{B(x^*,r)}f\|_{W_2^{k+i}(\mathbb{R}^{d-1})} \leqslant \mathcal{C}_i \|f\|_{W_2^{k+i}(B(x^*,r))}, \quad \text{for all } f \in \widetilde{W}_2^{k+i}(B(x^*,r)),$$

for $i \in \{0,1\}$, where the constants $\mathcal{C}_0$ and $\mathcal{C}_1$ are independent of x^* and r. Let $A_i = \widetilde{W}_2^{k+i}(B(x,r))$ and $B_i = W_2^{k+i}(\mathbb{R}^{d-1})$, $i \in \{0,1\}$, then the result follows from the operator interpolation property, which shows that $\widetilde{\mathcal{C}}_{\text{ext}}^{(\tau)} = \mathcal{C}_0^{1-\tau}\mathcal{C}_1^{\tau}$. $\square$

We can now turn attention to the spherical setting and we begin by defining the appropriate fractional subspace.

Definition 5.18. Let $\tau \in (0,1)$ and $k > \frac{d-1}{2}$ be a positive integer. Let $\Xi = \{\xi_i\}_{i=1}^N$ denote a set of distinct points in $G(z,\theta) \subset S^{d-1}$, then we define the local fractional order Sobolev subspace as

$$\widetilde{W}_2^{k+\tau}(G(z,\theta)) = \{f \in W_2^{k+\tau}(G(z,\theta)) : f(\xi) = 0,\ \xi \in \Xi \cap G(z,\theta)\}. \tag{5.29}$$

Using Proposition 5.16, we can recast Theorem 4.3 in terms of fractional Sobolev spaces, and its proof is completely analogous. In particular, following the proof through to inequality (4.7), we have

$$\|E_{G(z,\theta)}f\|^2_{W_2^{k+\tau}(S^{d-1})} \leqslant 4\mathcal{K}_\chi\mathcal{K}_\mathcal{A} \sum_{i=1}^{2} \|E_{B(x_i,r_i)}f_i\|^2_{W_2^{k+\tau}(\mathbb{R}^{d-1})}, \tag{5.30}$$

where

$$f_i = (\chi_i f)\circ\phi_i^{-1}|_{B(x_i,r_i)} \in W_2^{k+\tau}(B(x_i,r_i)), \quad \text{for } i \in \{1,2\}. \tag{5.31}$$

Let δ_k be as in Lemma 5.3. Let C_0 and c_0 be as in (2.16), and set

$$\mathcal{R}_0 = \max\left\{3, \frac{4C_0}{c_0\delta_k}\right\}. \tag{5.32}$$

Theorem 5.19. *Let $\tau \in (0,1)$ and $k > \frac{d-1}{2}$ be a positive integer. Let Ξ denote a set of distinct points on S^{d-1} whose mesh norm h satisfies*

(i) $h \in (0, \pi/6)$, (ii) $\mathcal{R}_0 h < C_{\mathcal{A}}/3$,

where $\mathcal{R}_0$ is given by (5.32). *Let $z \in S^{d-1}$, $\theta \in (\mathcal{R}_0 h, C_{\mathcal{A}}/3)$, and let $E_{G(z,\theta)}$ denote the continuous extension operator given by* (4.5). *Then there exists a constant $\widetilde{\mathcal{K}}^{(\tau)} > 0$, independent of z and θ, such that*

$$||E_{G(z,\theta)} f||_{W_2^{k+\tau}(S^{d-1})} \leqslant \widetilde{\mathcal{K}}^{(\tau)} ||f||_{W_2^{k+\tau}(G(z,\theta))} \quad f \in \widetilde{W}_2^{k+\tau}(G(z,\theta)). \tag{5.33}$$

Proof. The conditions of the theorem guarantee that the mesh norms $\{\rho_i\}_{i=1}^2$ of the transformed point sets $\{X_\theta^{(i)}\}_{i=1}^2$ satisfy (5.27). Since each f_i vanishes at $X_\theta^{(i)}$, $i \in \{1,2\}$, we can use Proposition 5.17, to continue inequality (5.30) as follows

$$\begin{aligned} ||E_{G(z,\theta)} f||^2_{W_2^{k+\tau}(S^{d-1})} &\leqslant 4\mathcal{K}_\chi \mathcal{K}_{\mathcal{A}} (\widetilde{\mathcal{C}}_{\text{ext}}^{(\tau)})^2 \sum_{i=1}^{2} ||f_i||^2_{W_2^{k+\tau}(B(x_i,r_i))} \\ &= (\widetilde{\mathcal{K}}^{(\tau)})^2 ||f||^2_{W_2^{k+\tau}(G(z,\theta))}, \end{aligned}$$

where $\widetilde{\mathcal{K}}^{(\tau)} = \sqrt{4\mathcal{K}_\chi \mathcal{K}_{\mathcal{A}}} \widetilde{\mathcal{C}}_{\text{ext}}^{(\tau)}$ is independent of both z and θ. □

Acknowledgments

Simon Hubbert and Tanya M. Morton thank Michael Johnson for his helpful email regarding Section 5. S.H. thanks Brad Baxter for his support and the T.M.M. thanks Larry Schumaker and Mike Neamtu.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] J.P. Aubin, Approximation of Elliptic Boundary-value Problems, Wiley-Interscience, New York, 1972.
[3] J. Duchon, Sur l'erreur d'interpolation des fonctions de plusieurs variable par les D^m-splines, RAIRO Anal. Numer. 12 (1978) 325–334.
[4] S. Hubbert, Radial basis function interpolation on the sphere, Ph.D. Thesis, Imperial College London, 2002.
[5] S. Hubbert, T.M. Morton, L_p-error estimates for radial basis function interpolation on the sphere, J. Approx. Theory 129 (2004) 58–77.

[6] W.A. Light, H. Wayne, Power functions and error estimates for radial basis function interpolation, J. Approx. Theory 92 (1992) 245–267.
[7] J.L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, Vol. 1, Springer, Berlin, Heidelberg, 1972.
[8] J.G. Ratcliffe, Foundations of Hyperbolic Manifolds, Springer, New York, 1994.
[9] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, NJ, 1971.
[10] H. Triebel, Interpolation Theory, Function Spaces and Differential Operators, North-Holland, Amsterdam, 1978.
[11] H. Wendland, Sobolev-type error estimates for interpolation by radial basis functions, in: A. Le Méhauté, C. Rabut, L.L. Schumaker (Eds.), Surface Fitting and Multiresolution Methods, Vanderbilt Univ. Press, Nashville, 1997, pp. 337–344.